PART B

Unit - VII

Numerical Methods - 3

7.1 Introduction

We have already discussed (*Unit* - III) theoretical solution of three important partial differential equations (p.d.es) namely, one dimensional wave equation, one dimensional heat equation and two dimensional Laplace's equation subjected to certain given conditions. They are referred to as Boundary Value Problems (B.V.Ps).

In this unit we discuss numerical solution of these p.d.es.

7.2 Classification of PDE s of second order

The general second order linear PDE in two independent variables x, y is of the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

where A, B, C, D, E, F are in general functions of x and y.

This equation is said to be

(1) *Parabolic* if $B^2 - 4AC = 0$

(2) *Elliptic* if $B^2 - 4AC < 0$

(3) *Hyperbolic* if $B^2 - 4AC > 0$

Now let us examine the nature of three PDE s which are under our discussion.

	P.D.E	A	B	C	$B^2 - 4AC$	Nature of the PDE
1.	One dimensional wave equation : $c^2 u_{xx} - u_{tt} = 0$	c^2	0	-1	$4c^2 > 0$	Hyperbolic
2.	One dimensional heat equation : $c^2 u_{xx} - u_t = 0$	c^2	0	0	0	Parabolic
3.	Two dimensional Laplace's equation : $u_{xx} + u_{yy} = 0$	1	0	1	$-4 < 0$	Elliptic

7.3 Finite difference approximation to ordinary and partial derivatives

Let $y = y(x)$ and its derivatives be single valued continuous functions of x. We have by Taylor's expansion,

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2!}\,y''(x) + \frac{h^3}{3!}\,y'''(x) + \cdots \qquad \ldots(1)$$

$$y(x-h) = y(x) - h\,y'(x) + \frac{h^2}{2!}\,y''(x) - \frac{h^3}{3!}\,y'''(x) + \cdots \qquad \ldots(2)$$

Assuming h to be small we shall neglect terms containing $h^2, h^3, \cdots$ in (1) and (2) so that we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} \text{ from (1) } \& \ y'(x) = \frac{y(x) - y(x-h)}{h} \text{ from (2).}$$

Also we have by (1) – (2) $y'(x) = \dfrac{y(x+h) - y(x-h)}{2h}$

These three expressions for $y'(x)$ are the finite difference approximation in terms of forward, backward and and central difference respectively.

Also (1) + (2) by neglecting terms containing $h^3, h^4 \cdots$ will give us,

$$y(x+h) + y(x-h) = 2\,y(x) + h^2\,y''(x)$$

or $$y''(x) = \frac{y(x+h) - 2\,y(x) + y(x-h)}{h^2}$$

This is the finite difference approximation for $y''(x)$.

Extension to the partial derivatives

Let $u = u(x, y)$ be a function of two independent variables. The finite difference approximation for the first order partial derivatives : u_x, u_y and second order partial derivatives : u_{xx}, y_{yy} are as follows.

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} \qquad \ldots(1)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x, y) - u(x-h, y)}{h} \qquad \ldots(2)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x-h, y)}{2\,h} \qquad \ldots(3)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{u(x, y+k) - u(x, y)}{k} \qquad \ldots (4)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{u(x, y) - u(x, y-k)}{k} \qquad \ldots (5)$$

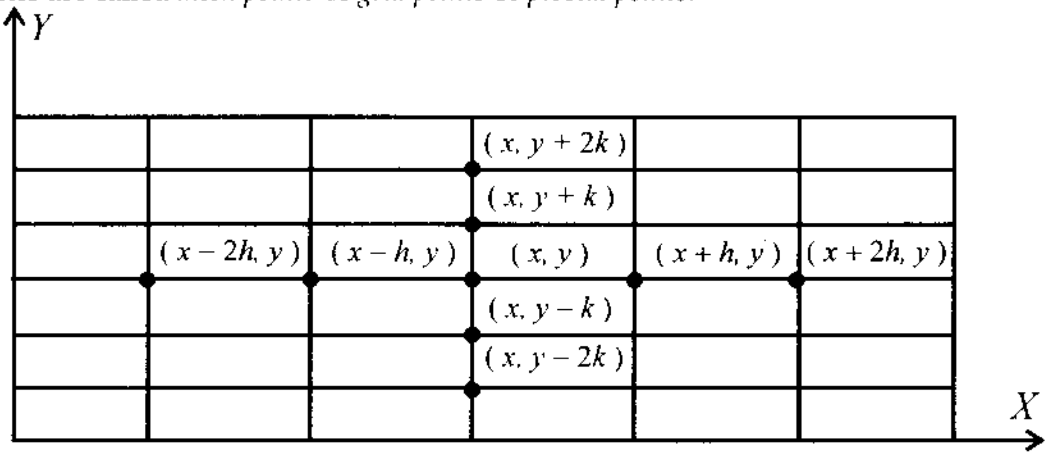
$$u_y = \frac{\partial u}{\partial y} = \frac{u(x, y+k) - u(x, y-k)}{2k} \qquad \ldots (6)$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\left[u(x+h, y) - 2u(x, y) + u(x-h, y) \right] \qquad \ldots (7)$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}\left[u(x, y+k) - 2u(x, y) + u(x, y-k) \right] \qquad \ldots (8)$$

7.4 Numerical solution of a PDE

Consider a rectangular region R in the $x-y$ plane. Let us divide this region into a network of rectangles of sides h and k. In other words, we draw lines $x = ih$, $y = jk$; $i, j = 1, 2, 3, \cdots$ being parallel to the Y - axis and X axis respectively resulting into a network of rectangles. The points of intersection of these lines are called *mesh points* or *grid points* or *pivotal points.*



We write $u(x, y) = u(ih, jk)$ and the finite difference approximation for the partial derivatives given by (1) to (8) are put in the following modified form.

$$u_x = \frac{1}{h}\left[u_{i+1, j} - u_{i, j} \right] \qquad \ldots (F_1)$$

$$u_x = \frac{1}{h}\left[u_{i, j} - u_{i-1, j} \right] \qquad \ldots (F_2)$$

$$u_x = \frac{1}{2h}\left[u_{i+1, j} - u_{i-1, j} \right] \qquad \ldots (F_3)$$

$$u_y = \frac{1}{k}\left[u_{i,\,j+1} - u_{i,\,j}\right] \qquad \ldots (F_4)$$

$$u_y = \frac{1}{k}\left[u_{i,\,j} - u_{i,\,j-1}\right] \qquad \ldots (F_5)$$

$$u_y = \frac{1}{2k}\left[u_{i,\,j+1} - u_{i,\,j-1}\right] \qquad \ldots (F_6)$$

$$u_{xx} = \frac{1}{h^2}\left[u_{i-1,\,j} - 2u_{i,\,j} + u_{i+1,\,j}\right] \qquad \ldots (F_7)$$

$$u_{yy} = \frac{1}{k^2}\left[u_{i,\,j-1} - 2u_{i,\,j} + u_{i,\,j+1}\right] \qquad \ldots (F_8)$$

The substitution of these finite difference approximations into the given PDE converts (*approximates*) the PDE into a finite difference equation. We solve this equation under the given set of conditions. Any condition involving partial derivative is also approximated interms of finite differences appropriately.

This will enable us to determine $u_{i,\,j}$ explicitly at all the interior mesh points. These values constitute a numerical solution to the given PDE.

We discuss numerical solution of the three important PDE s namely

(1) One dimensional wave equation

(2) One dimensional heat equation

(3) Two dimensional Laplace equation.

7.41 Numerical solution of the one dimensional wave equation

We seek the numerical solution of the wave equation

$$c^2 u_{xx} = u_{tt} \qquad \ldots (1)$$

subject to the boundary conditions

$$u(0,\ t) = 0 \qquad \ldots (2)$$

$$u(l,\ t) = 0 \qquad \ldots (3)$$

and the initial conditions

$$u(x,\ 0) = f(x) \qquad \ldots (4)$$

$$u_t(x,\ 0) = 0 \qquad \ldots (5)$$

We shall substitute the finite difference approximation for the partial derivatives present in (1). [F_7 *and* F_8 *are to be used*]

$$\therefore \quad c^2 \cdot \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] = \frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right]$$

or $$c^2\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] = \frac{h^2}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right]$$

Taking $k/h = \lambda$ we have,

$$c^2\lambda^2\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] = \left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right]$$

$$\therefore \quad u_{i,j+1} = 2(1 - c^2\lambda^2)u_{i,j} + c^2\lambda^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad \ldots (6)$$

For convenience let us choose λ such that $1 - c^2\lambda^2 = 0$

i.e., $1 - c^2(k^2/h^2) = 0$ or $k^2 = h^2/c^2 \Rightarrow k = h/c$

Thus (6) reduces to the form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \ldots (7)$$

This is called the ***explicit formula*** for the solution of the wave equation.

Further we express the initial condition (4) involving partial derivative w.r.t. t in terms of finite differences. We consider

$$u_t = \frac{1}{2k}\left[u_{i,j+1} - u_{i,j-1}\right] \quad (By\ F_6)$$

$$u_t(x,0) = 0 \text{ gives } \frac{1}{2k}\left[u_{i,1} - u_{i,-1}\right] = 0 \quad (Taking\ j = 0)$$

or $$u_{i,1} = u_{i,-1} \qquad \ldots (8)$$

Putting $j = 0$ in (7) we get

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

i.e., $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$ by using (8).

or $$2u_{i,1} = u_{i-1,0} + u_{i+1,0}$$

$$\therefore \quad u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right] \qquad \ldots (9)$$

Further $u(0,t) = 0$ and $u(l,t) = 0$ means $u_{0,j} = 0$ and $u_{l,j} = 0$ which implies that the values along the first column and last column are zero.

$u(x,0) = f(x)$ means $u_{i,0} = f(x)$ give the values of u along the first row.

Finally $u_t(x, 0) = 0$ modified into the form (9) giving $u_{i,1}$ will give us the values of u along the second row. These values will help us to obtain the rest of the value of u at the mesh points by the explicit formula (7).

Thus we are able to determine $u(x, t)$ at all interior mesh points.

Working procedure for problems

- Based on the given step sizes h and k, we form the rectangular network of sides h and k.
- Given h only we can get $k = h/c$. The points of division of x are $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots x_n = x_0 + nh$ and that of t are $t_0, t_1 = t_0 + k, t_2 = t_0 + 2k, \cdots t_n = t_0 + nk$.
- We form the basic informative table :

t \ x		x_0	x_1	x_2	...	x_n
		0	1	2	...	n
t_0	0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	...	$u_{n,0}$
t_1	1	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	...	$u_{n,1}$
t_2	2	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	...	$u_{n,2}$
...	...	...	...	...	...	...
t_n	n	$u_{0,n}$	$u_{1,n}$	$u_{2,n}$	...	$u_{n,n}$

- We instantly write the value of $u_{i,j}$ with reference to the conditions $u(0, t) = 0$ and $u(l, t) = 0$
  [*Values along the first and last column are zero*]
- The values along the first row are obtained by direct computation using the condition $u(x, 0) = f(x)$
- The values along the second row are obtained by the relation $u_{i,1} = \frac{1}{2} \cdot \left[u_{i-1,0} + u_{i+1,0} \right]$

Rest of the values are found from the relation

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

WORKED PROBLEMS

1. *Solve the wave equation* $\dfrac{\partial^2 u}{\partial t^2} = 4\,\dfrac{\partial^2 u}{\partial x^2}$ *subject to* $u = (0, t) = 0,\ u(4, t) = 0,$ $u_t(x, 0) = 0$ *and* $u(x, 0) = x(4-x)$ *by taking* $h = 1,\ k = 0.5$ *upto four steps.*

>> Step size of x : $h = 1$; Step size of t : $k = 0.5$

Since $0 \le x \le 4$, the points of division are 0, 1, 2, 3, 4. Since $k = 0.5$, the values corresponding to t upto four steps are 0, 0.5, 1, 1.5 and 2. We have the following initial table. The values in the first and last column are zero since $u(0, t) = 0 = u(4, t)$

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0} = 0$
t_1	0.5	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1} = 0$
t_2	1	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2} = 0$
t_3	1.5	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3} = 0$
t_4	2	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4} = 0$

Now consider $u(x, 0) = x(4-x)$

$$\therefore \quad u_{1,0} = u(1, 0) = 3\ ;\ u_{2,0} = u(2, 0) = 4\ ;\ u_{3,0} = 3$$

(The first row in the table is completed)

Next consider $u_{i,1} = \dfrac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$$\therefore \quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+4) = 2$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(3+3) = 3$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(4+0) = 2$$

(The second row in the table is completed)

We now consider the explicit formula to find the remaining values in the table.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$$u_{1,\,2} = u_{0,\,1} + u_{2,\,1} - u_{1,\,0} = 0 + 3 - 3 = 0$$

$$u_{2,\,2} = u_{1,\,1} + u_{3,\,1} - u_{2,\,0} = 2 + 2 - 4 = 0$$

$$u_{3,\,2} = u_{2,\,1} + u_{4,\,1} - u_{3,\,0} = 3 + 0 - 3 = 0$$

(The third row is completed)

$$u_{1,\,3} = u_{0,\,2} + u_{2,\,2} - u_{1,\,1} = 0 + 0 - 2 = -2$$

$$u_{2,\,3} = u_{1,\,2} + u_{3,\,2} - u_{2,\,1} = 0 + 0 - 3 = -3$$

$$u_{3,\,3} = u_{2,\,2} + u_{4,\,2} - u_{3,\,1} = 0 + 0 - 2 = -2$$

(The fourth row is completed)

$$u_{1,\,4} = u_{0,\,3} + u_{2,\,3} - u_{1,\,2} = 0 - 3 - 0 = -3$$

$$u_{2,\,4} = u_{1,\,3} + u_{3,\,3} - u_{2,\,2} = -2 - 2 - 0 = -4$$

$$u_{3,\,4} = u_{2,\,3} + u_{4,\,3} - u_{3,\,2} = -3 + 0 - 0 = -3$$

(The last row is completed)

Thus the required values of $u_{i,\,j}$ are tabulated.

t \ x	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	− 2	− 3	− 2	0
2	0	− 3	− 4	− 3	0

2. *Solve numerically* $u_{xx} = 0.0625\, u_{tt}$ *subject to the conditions* $u(0,\, t) = 0 = u(5,\, t)$ $u(x,\, 0) = x^2(x-5)$ *and* $u_t(x,\, 0) = 0$ *by taking* $h = 1$ *for* $0 \le t \le 1$

>> The wave equation in the standard form is $c^2 u_{xx} = u_{tt}$ and hence the given equation be put in the form $(1/0.0625)\, u_{xx} = u_{tt}$

That is, $16\, u_{xx} = u_{tt}$ where $c^2 = 16$ or $c = 4$

Since $h = 1$, we have $k = h/c = 1/4 = 0.25$

Step size of x : $h = 1$ where $0 \le x \le 5$

Step size of t : $k = 0.25$ where $0 \le t \le 1$ as required.

$\therefore$ values of x are 0, 1, 2, 3, 4, 5

values of t are 0, 0.25, 0.5, 0.75, 1

We have the following initial table. The values in the first and last column are zero since $u(0, t) = 0 = u(5, t)$

t \ x		x_0 = 0	x_1 = 1	x_2 = 2	x_3 = 3	x_4 = 4	x_5 = 5
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0} = 0$
t_1	0.25	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1} = 0$
t_2	0.5	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2} = 0$
t_3	0.75	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3} = 0$
t_4	1	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4} = 0$

Now consider $u(x, 0) = x^2(x-5)$

$\therefore \quad u_{1,0} = u(1, 0) = -4 \;;\; u_{2,0} = -12 \;;\; u_{3,0} = -18 \;;\; u_{4,0} = -16$

(First row in the table is completed)

Next consider $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$$\therefore \quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0-12) = -6$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(-4-18) = -11$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(-12-16) = -14$$

$$u_{4,1} = \frac{1}{2}\left[u_{3,0} + u_{5,0}\right] = \frac{1}{2}(-18+0) = -9$$

(Second row in the table is completed)

We now consider the explicit formula to find the remaining values in the table.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$$\therefore \quad u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 - 11 + 4 = -7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = -6 - 14 + 12 = -8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = -11 - 9 + 18 = -2$$

$u_{4,\,2} = u_{3,\,1} + u_{5,\,1} - u_{4,\,0} = -14 + 0 + 16 = 2$

(Third row in the table is completed)

$u_{1,\,3} = u_{0,\,2} + u_{2,\,2} - u_{1,\,1} = 0 - 8 + 6 = -2$

$u_{2,\,3} = u_{1,\,2} + u_{3,\,2} - u_{2,\,1} = -7 - 2 + 11 = 2$

$u_{3,\,3} = u_{2,\,2} + u_{4,\,2} - u_{3,\,1} = -8 + 2 + 14 = 8$

$u_{4,\,3} = u_{3,\,2} + u_{5,\,2} - u_{4,\,1} = -2 + 0 + 9 = 7$

(Fourth row in the table is completed)

$u_{1,\,4} = u_{0,\,3} + u_{2,\,3} - u_{1,\,2} = 0 + 2 + 7 = 9$

$u_{2,\,4} = u_{1,\,3} + u_{3,\,3} - u_{2,\,2} = -2 + 8 + 8 = 14$

$u_{3,\,4} = u_{2,\,3} + u_{4,\,3} - u_{3,\,2} = 2 + 7 + 2 = 11$

$u_{4,\,4} = u_{3,\,3} + u_{5,\,3} - u_{4,\,2} = 8 + 0 - 2 = 6$

(Fifth row in the table is completed)

Thus the required values of $u_{i,\,j}$ are tabulated.

t \ x	0	1	2	3	4	5
0	0	**− 4**	**− 12**	**− 18**	**− 16**	0
0.25	0	**− 6**	**− 11**	**− 14**	**− 9**	0
0.5	0	**− 7**	**− 8**	**− 2**	**2**	0
0.75	0	**− 2**	**2**	**8**	**7**	0
1	0	**9**	**14**	**11**	**6**	0

3. *Solve* $25\,u_{xx} = u_{tt}$ *at the pivotal points given* $u(0, t) = 0 = u(5, t)$; $u_t(x, 0) = 0$ *and* $u(x, 0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$ *by taking* $h = 1$ *Compute* $u(x, t)$ *for* $0 \le t \le 0.1$

>> Comparing the wave equation $c^2 u_{xx} = u_{tt}$ with the given equation $25\,u_{xx} = u_{tt}$, we have $c^2 = 25$ or $c = 5$. Also $k = h/c = 1/5 = 0.2$

Since $h = 1$, the values of x in $0 \le x \le 5$ are 0, 1, 2, 3, 4, 5 and the values of t are 0, 0.2, 0.4, 0.6, 0.8, 1

The initial table showing the values given and the values to be computed is as follows.

t \ x		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	$u_{0,0}=0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0}=0$
t_1	0.2	$u_{0,1}=0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1}=0$
t_2	0.4	$u_{0,2}=0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2}=0$
t_3	0.6	$u_{0,3}=0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3}=0$
t_4	0.8	$u_{0,4}=0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4}=0$
t_5	1	$u_{0,5}=0$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5}$	$u_{5,5}=0$

We have $u(x,0)=\begin{cases}20x, & 0\le x\le 1\\ 5(5-x), & 1\le x\le 5\end{cases}$

$\therefore \quad u_{1,0}=u(1,0)=20\ ;\ u_{2,0}=u(2,0)=5\times 3=15\ ;$

$u_{3,0}=u(3,0)=5\times 2=10\ ;\ u_{4,0}=u(4,0)=5$

(First row in the table is completed)

Next consider $u_{i,1}=\frac{1}{2}\left[u_{i-1,0}+u_{i+1,0}\right]$

$$\therefore \quad u_{1,1}=\frac{1}{2}\left[u_{0,0}+u_{2,0}\right]=\frac{1}{2}(0+15)=7.5$$

$$u_{2,1}=\frac{1}{2}\left[u_{1,0}+u_{3,0}\right]=\frac{1}{2}(20+10)=15$$

$$u_{3,1}=\frac{1}{2}\left[u_{2,0}+u_{4,0}\right]=\frac{1}{2}(15+5)=10$$

$$u_{4,1}=\frac{1}{2}\left[u_{3,0}+u_{5,0}\right]=\frac{1}{2}(10+0)=5$$

(Second row in the table is completed)

We now consider the explicit formula to find the remaining values in the table.

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1}$$

$$\therefore \quad u_{1,2}=u_{0,1}+u_{2,1}-u_{1,0}=0+15-20=-5$$

$$u_{2,2}=u_{1,1}+u_{3,1}-u_{2,0}=7.5+10-15=2.5$$

$$u_{3,2}=u_{2,1}+u_{4,1}-u_{3,0}=15+5-10=10$$

$$u_{4,2}=u_{3,1}+u_{5,1}-u_{4,0}=10+0-5=5$$

(Third row in the table is completed)

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$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 2.5 - 7.5 = -5$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = -5 + 10 - 15 = -10$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 2.5 + 5 - 10 = -2.5$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 10 + 0 - 5 = 5$$

(*Fourth row in the table is completed*)

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 10 + 5 = -5$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -5 - 2.5 - 2.5 = -10$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -10 + 5 - 10 = -15$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -2.5 + 0 - 5 = -7.5$$

(*Fifth row in the table is completed*)

$$u_{1,5} = u_{0,4} + u_{2,4} - u_{1,3} = 0 - 10 + 5 = -5$$

$$u_{2,5} = u_{1,4} + u_{3,4} - u_{2,3} = -5 - 15 + 10 = -10$$

$$u_{3,5} = u_{2,4} + u_{4,4} - u_{3,3} = -10 - 7.5 + 2.5 = -15$$

$$u_{4,5} = u_{3,4} + u_{5,4} - u_{4,3} = -15 + 0 - 5 = -20$$

(*Last row in the table is completed*)

Thus the required values of $u_{i,j}$ **are tabulated**.

t \ x	0	1	2	3	4	5
0	0	**20**	**15**	**10**	**5**	0
0.2	0	**7.5**	**15**	**10**	**5**	0
0.4	0	**– 5**	**2.5**	**10**	**5**	0
0.6	0	**– 5**	**– 10**	**– 2.5**	**5**	0
0.8	0	**– 5**	**– 10**	**– 15**	**– 7.5**	0
1	0	**– 5**	**– 10**	**– 15**	**– 20**	0

4. *Solve* : $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ *given that* $u(x, 0) = 0$; $u(0, t) = 0$; $u_t(x, 0) = 0$ *and* $u(1, t) = 100 \sin(\pi t)$ *in the range* $0 \le t \le 1$ *by taking* $h = 1/4$

>> [*It may be noticed that two of the conditions are different compared to the earlier three problems. The approach for solving this problem continues to be the same*]

Comparing the wave equation $c^2 u_{xx} = u_{tt}$ with the given equation $u_{xx} = u_{tt}$, we have $c^2 = 1$ or $c = 1$

By data $h = 1/4$, $k = h/c = 1/4$

The points of division of x (*in* $0 \le x \le 1$) as well as t are 0, 1/4, 1/2, 3/4, 1.

The condition $u(x, 0) = 0$ means that the value of u along the first row are zero and $u(0, t) = 0$ means that the value of u along the first column are zero. The initial table showing the values given and the values to be computed is as follows.

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0$	$u_{2,0} = 0$	$u_{3,0} = 0$	$u_{4,0} = 0$
t_1	1/4	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$
t_2	1/2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$
t_3	3/4	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$
t_4	1	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$

We have seen that the condition $u_t(x, 0) = 0$ will lead us to the formula

$$u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$$

$$\therefore \quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+0) = 0$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(0+0) = 0$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(0+0) = 0$$

$u_{4,1} = \dfrac{1}{2}\left[u_{3,0} + u_{5,0}\right]$ and we do not have $u_{5,0}$ on hand.

Hence $u_{4,1} = u(x_4, t_1) = u(1, 1/4) = 100 \sin(\pi/4) = 70.7$

(*Second row of the table is completed*)

We shall now consider the explicit formula to find the remaining values in the table.

$$u_{i,\,j+1} = u_{i-1,\,j} + u_{i+1,\,j} - u_{i,\,j-1}$$

$\therefore$ $u_{1,\,2} = u_{0,\,1} + u_{2,\,1} - u_{1,\,0} = 0$

$u_{2,\,2} = u_{1,\,1} + u_{3,\,1} - u_{2,\,0} = 0$

$u_{3,\,2} = u_{2,\,1} + u_{4,\,1} - u_{3,\,0} = 70.7$

$u_{4,\,2} = u_{3,\,1} + u_{5,\,1} - u_{4,\,0}$ is inapplicable.

Hence $u_{4,\,2} = u(x_4,\ t_2) = u(1,\ 1/2) = 100 \sin(\pi/2) = 100$

(*Third row of the table is completed*)

$u_{1,\,3} = u_{0,\,2} + u_{2,\,2} - u_{1,\,1} = 0$

$u_{2,\,3} = u_{1,\,2} + u_{3,\,2} - u_{2,\,1} = 70.7$

$u_{3,\,3} = u_{2,\,2} + u_{4,\,2} - u_{3,\,1} = 100$

$u_{4,\,3} = u(x_4,\ t_3) = u(1,\ 3/4) = 100 \sin(3\pi/4) = 70.7$

(*Fourth row of the table is completed*)

$u_{1,\,4} = u_{0,\,3} + u_{2,\,3} - u_{1,\,2} = 70.7$

$u_{2,\,4} = u_{1,\,3} + u_{3,\,3} - u_{2,\,2} = 100$

$u_{3,\,4} = u_{2,\,3} + u_{4,\,3} - u_{3,\,2} = 70.7$

$u_{4,\,4} = u(x_4,\ t_4) = u(1,\ 1) = 100 \sin(\pi) = 0$

(*Last row of the table is completed*)

Thus the required values of $u_{i,\,j}$ are tabulated

t \ x	0	1/4	1/2	3/4	1
0	0	0	0	0	0
1/4	0	**0**	**0**	**0**	**70.7**
1/2	0	**0**	**0**	**70.7**	**100**
3/4	0	**0**	**70.7**	**100**	**70.7**
1	0	**70.7**	**100**	**70.7**	**0**

5. *Solve the wave equation* $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ *given* $u(0,t) = u(5,t) = 0,\ t \geq 0$

$$u(x,0) = x(5-x),\ \frac{\partial}{\partial t}u(x,0) = 0,\ 0 < x < 5$$

Find u *at* $t = 2$ *given* $h = 1,\ k = 0.5$

>> Here $c^2 = 4$ or $c = 2$ and $k = h/c$

We have the explicit formula for the solution of the wave equation given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \ldots (1)$$

Also the condition $\frac{\partial}{\partial t}u(x,0) = 0$ will give the formula from (1) in the form,

$$u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right] \quad \ldots (2)$$

Since $h = 1$, values of x are $0, 1, 2, 3, 4, 5$

Since $k = 0.5$, values of t are $0, 0.5, 1, 1.5, 2$

$$u(0,t) = 0 \Rightarrow u_{0,0} = 0 = u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4}$$

$$u(5,t) = 0 \Rightarrow u_{5,0} = 0 = u_{5,1} = u_{5,2} = u_{5,3} = u_{5,4}$$

(*The values along the first and last column are zero*)

We have the following initial table.

t \ x		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0} = 0$
t_1	0.5	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1} = 0$
t_2	1	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2} = 0$
t_3	1.5	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3} = 0$
t_4	2	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4} = 0$

$$u(x,0) = x(5-x)$$

$\therefore \quad u(1,0) = u_{1,0} = 4$; Also $u_{2,0} = 6,\ u_{3,0} = 6,\ u_{4,0} = 4$

(*The first rwo in the table is completed*)

Next we have from (2),

$$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+6) = 3$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(4+6) = 5$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(6+4) = 5$$

$$u_{4,1} = \frac{1}{2}\left[u_{3,0} + u_{5,0}\right] = \frac{1}{2}(6+0) = 3$$

(*The second row in the table is completed*)

Next we have from (1) the following.

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0+5-4 = 1$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 3+5-6 = 2$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 5+3-6 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 5+0-4 = 1$$

(*The third row in the table is completed*)

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0+2-3 = -1$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 1+2-5 = -2$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 2+1-5 = -2$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 2+0-3 = -1$$

(*The fourth row in the table is completed*)

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0-2-1 = -3$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -1-2-2 = -5$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -2-1-2 = -5$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -2+0-1 = -3$$

Thus the required values of u at $t = 2$ are $-3, -5, -5, -3$

- -

6. *Solve the wave equation* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ *subject to the boundary conditions,* $u(0, t) = 0 = u(1, t)$, $t \geq 0$ *and the initial conditions,* $u(x, 0) = \sin \pi x$, $\dfrac{\partial u}{\partial t}(x, 0) = 0$, $0 < x < 1$ *by taking* $h = 1/4$ *and* $k = 1/5$. *Carryout second level solutions in the time scale.*

>> Here $h = 1/4$ and hence the values of x are $0, 1/4, 1/2, 3/4, 1$

Also $k = 1/5$ and we have $c^2 = 1$

Note : *Refer article 7.41 and retrace the steps of obtaining the following equation on substituting the finite difference approximation for the two partial derivatives in the wave equation.*

$$u_{i, j+1} = 2(1 - c^2\lambda^2)\, u_{i, j} + c^2\lambda^2\, (u_{i-1, j} + u_{i+1, j}) - u_{i, j-1} \qquad \ldots (1)$$

where $\lambda = k/h$

It should be noted that if $k = h/c$ the first term vanish and in the given problem $k \neq h/c$

Further the condition $\dfrac{\partial u}{\partial t}(x, 0) = 0$ gives $u_{i, 1} = u_{i, -1}$

Hence by putting $j = 0$ in (1) we have,

$$u_{i, 1} = 2(1 - c^2\lambda^2)\, u_{i, 0} + c^2\lambda^2 (u_{i-1, 0} + u_{i+1, 0}) - u_{i, -1}$$

But $\quad u_{i, -1} = u_{i, 1}$ and $c^2\lambda^2 = k^2/h^2 = 16/25$

$$\therefore \quad 2u_{i, 1} = \frac{18}{25} u_{i, 0} + \frac{16}{25}\left(u_{i-1, 0} + u_{i+1, 0}\right)$$

or

$$u_{i, 1} = \frac{1}{25}\left\{9u_{i, 0} + 8(u_{i-1, 0} + u_{i+1, 0})\right\} \qquad \ldots (2)$$

Let us consider the given conditions.

$$u(0, t) = 0 \Rightarrow u_{0, 0} = u_{0, 1} = u_{0, 2} = u_{0, 3} = \cdots = 0$$

The values of x in $0 < x < 1$ with step size $h = 1/4$ are

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1/2, \quad x_3 = 3/4 \text{ and } x_4 = 1$$

$$u(1, t) = 0 \Rightarrow u_{4, 0} = u_{4, 1} = u_{4, 2} = \cdots = 0$$

Also we have $u(x, 0) = \sin \pi x$

$$u(x_1, 0) = u_{1, 0} = u(1/4, 0) = \sin(\pi/4) = 0.7071$$

$$u(x_2, 0) = u_{2,0} = u(1/2, 0) = \sin(\pi/2) = 1$$

$$u(x_3, 0) = u_{3,0} = u(3/4, 0) = \sin(3\pi/4) = 0.7071$$

$$u(x_4, 0) = u_{4,0} = u(1, 0) = \sin \pi = 0$$

Next we shall consider (2) and compute $u_{1,1}$, $u_{2,1}$, $u_{3,1}$ which represents the first level solution in the time scale.

$$u_{1,1} = \frac{1}{25}\left[9u_{1,0} + 8(u_{0,0} + u_{2,0})\right]$$

$$= \frac{1}{25}\left[9(0.7071) + 8(0+1)\right] = 0.574556$$

$$u_{2,1} = \frac{1}{25}\left[9u_{2,0} + 8(u_{1,0} + u_{3,0})\right]$$

$$= \frac{1}{25}\left[9(1) + 8(0.7071 + 0.7071)\right] = 0.812544$$

$$u_{3,1} = \frac{1}{25}\left[9u_{3,0} + 8(u_{2,0} + u_{4,0})\right]$$

$$= \frac{1}{25}\left[9(0.7071) + 8(1+0)\right] = 0.574556$$

Also from (1) when $j = 1$ we have

$$u_{i,2} = \frac{18}{25}u_{i,1} + \frac{16}{25}\left(u_{i-1,1} + u_{i+1,1}\right) - u_{i,0}$$

i.e., $$u_{i,2} = \frac{2}{25}\left[9u_{i,1} + 8(u_{i-1,1} + u_{i+1,1})\right] - u_{i,0}$$

We shall compute $u_{1,2}$, $u_{2,2}$, $u_{3,2}$

$$u_{1,2} = \frac{2}{25}\left[9(0.574556) + 8(0 + 0.812544)\right] - 0.7071 = 0.226608$$

$$u_{2,2} = \frac{2}{25}\left[9(0.812544) + 8(0.574556 + 0.574556)\right] - 1 = 0.32046$$

$$u_{3,2} = \frac{2}{25}\left[9(0.574556) + 8(0.812544 + 0)\right] - 0.7071 = 0.226608$$

The solution upto second level correct to four decimal places are tabulated.

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	0	**0.7071**	**1**	**0.7071**	0
t_1	1/5	0	**0.5746**	**0.8125**	**0.5746**	0
t_2	2/5	0	**0.2266**	**0.3205**	**0.2266**	0

7. *Solve* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ *with the boundary conditions* $u(0, t) = 0 = u(1, t),\ t \geq 0$ *and initial conditions* $u(x, 0) = \sin \pi x,\ \dfrac{\partial u}{\partial t}(x, 0) = 0,\ 0 < x < 1$ *taking* $h = k = 0.2$ *at* $t = 1.0$

>> We note that $c^2 = 1$ or $c = 1$ and $k = h/c$

We have the explicit formula

$$u_{i,\,j+1} = u_{i-1,\,j} + u_{i+1,\,j} - u_{i,\,j-1} \qquad \ldots (1)$$

Further $\dfrac{\partial u}{\partial t}(x, 0) = 0$ will give us

$$u_{i,\,1} = \frac{1}{2}\left[u_{i-1,\,0} + u_{i+1,\,0}\right] \qquad \ldots (2)$$

The points of x and t in stepsize of 0.2 are 0, 0.2, 0.4, 0.6, 0.8 and 1

$u(0, t) = 0$ and $u(1, t) = 0$ will give us

$$u_{0,\,0} = 0 = u_{0,\,1} = u_{0,\,2} = u_{0,\,3} = u_{0,\,4} = u_{0,\,5}$$

$$u_{5,\,0} = 0 = u_{5,\,1} = u_{5,\,2} = u_{5,\,3} = u_{5,\,4} = u_{5,\,5}$$

Also we have $u(x, 0) = \sin \pi x$ by data.

$x_0 = 0,\ x_1 = 0.2 = 1/5,\ x_2 = 0.4 = 2/5,$
$x_3 = 0.6 = 3/5,\ x_4 = 0.8 = 4/5$ and $x_5 = 1$

$\therefore \quad u_{1,\,0} = \sin(\pi/5) = 0.5878\ ;\ u_{2,\,0} = \sin(2\pi/5) = 0.9511$

$u_{3,\,0} = \sin(3\pi/5) = 0.9511\ ;\ u_{4,\,0} = \sin(4\pi/5) = 0.5878$

Next we have from (2),

$$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = 0.4756$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = 0.7695$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = 0.7695$$

$$u_{4,1} = \frac{1}{2}\left[u_{3,0} + u_{5,0}\right] = 0.4756$$

Also we have from (1)

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

By taking $i = 1, 2, 3, 4$ we obtain

$$u_{1,2} = 0 + 0.7695 - 0.5878 = 0.1817$$
$$u_{2,2} = 0.4756 + 0.7695 - 0.9511 = 0.294$$
$$u_{3,2} = 0.7695 + 0.4756 - 0.9511 = 0.294$$
$$u_{4,2} = 0.7695 + 0 - 0.5878 = 0.1817$$

Again from (1)

$$u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$$

Taking $i = 1, 2, 3, 4$ we have

$$u_{1,3} = 0 + 0.294 - 0.4756 = -0.1816$$
$$u_{2,3} = 0.1817 + 0.294 - 0.7695 = -0.2938$$
$$u_{3,3} = 0.294 + 0.1817 - 0.7695 = -0.2938$$
$$u_{4,3} = 0.294 + 0 - 0.4756 = -0.1816$$

Again from (1)

$$u_{i,4} = u_{i-1,3} + u_{i+1,3} - u_{i,2}$$

We have as before

$$u_{1,4} = 0 - 0.2938 - 0.1817 = -0.4755$$
$$u_{2,4} = -0.1816 - 0.2938 - 0.294 = -0.7694$$
$$u_{3,4} = -0.2938 - 0.1816 - 0.294 = -0.7694$$
$$u_{4,4} = -0.2938 + 0 - 0.1817 = -0.4755$$

Again from (1)

$$u_{i,5} = u_{i-1,4} + u_{i+1,4} - u_{i,3}$$

We have as before

$$u_{1,5} = 0 - 0.7694 + 0.1816 = -0.5878$$

$$u_{2,5} = -0.4755 - 0.7694 + 0.2938 = -0.9511$$

$$u_{3,5} = -0.7694 - 0.4755 + 0.2938 = -0.9511$$

$$u_{4,5} = -0.7694 + 0 + 0.1816 = -0.5878$$

Thus the solution at $t = 1.0$ are

− 0.5878 , − 0.9511 , − 0.9511 , − 0.5878

7.42 Numerical solution of the one dimensional heat equation

We seek the numerical solution of the heat equation

$$u_t = c^2 u_{xx} \qquad \ldots (1)$$

subject to the boundary conditions

$$u(0, t) = 0 \qquad \ldots (2)$$

$$u(l, t) = 0 \qquad \ldots (3)$$

and the initial condition

$$u(x, 0) = f(x) \qquad \ldots (4)$$

We shall substitute the finite difference approximation for the partial derivatives present in (1). [F_4 *and* F_7 *be used*]

$$\therefore \quad \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = c^2 \cdot \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or $$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Taking $kc^2/h^2 = a$, the above equation becomes

$$u_{i,j+1} = u_{i,j} + a\,u_{i-1,j} - 2a\,u_{i,j} + a\,u_{i+1,j}$$

or $$u_{i,j+1} = a\,u_{i-1,j} + (1 - 2a)\,u_{i,j} + a\,u_{i+1,j} \qquad \ldots (5)$$

This is called ***Schmidt explicit formula*** valid for $0 < a \leq 1/2$

For convenience let us set $1 - 2a = 0$ or $a = 1/2$

That is $\frac{k c^2}{h^2} = \frac{1}{2}$ or $k = \frac{h^2}{2 c^2}$.

Thus (5) now becomes

$$u_{i, j+1} = \frac{1}{2} \left[u_{i-1, j} + u_{i+1, j} \right] \qquad \ldots (6)$$

This is called ***Bendre - Schmidt formula*** and we use this formula along with the given conditions to compute the values of u at the interior mesh points.

Note : ***The working procedure for problems is same as in the case of the wave equation.*** *Given h only we find $k = h^2/2c^2$. We prepare the initial table showing the values given and the values to be computed. After finding the values from the initial condition, we use the formula $u_{i, j+1} = \frac{1}{2} \cdot \left[u_{i-1, j} + u_{i+1, j} \right]$ to compute the rest of the required values in the table.*

WORKED PROBLEMS

8. *Find the numerical solution of the parabolic equation* $\frac{\partial^2 u}{\partial x^2} = 2 \frac{\partial u}{\partial t}$

   when $u(0, t) = 0 = u(4, t)$ and $u(x, 0) = x(4 - x)$ by taking $h = 1$. Find the values upto $t = 5$.

>> The standard form of the one dimensional heat equation is $u_t = c^2 u_{xx}$ and the given equation can be put in the form $u_t = (1/2) u_{xx}$

$\therefore \quad c^2 = 1/2$. Since $h = 1$, $k = h^2/2c^2 = 1$

The values of x in $0 \le x \le 4$ with $h = 1$ are 0, 1, 2, 3, 4 and the values of t with $k = 1$ are 0, 1, 2, 3, 4, 5.

We have the following initial table wherein the values in the first and last column are zero since $u(0, t) = 0$, $u(4, t) = 0$

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0, 0} = 0$	$u_{1, 0}$	$u_{2, 0}$	$u_{3, 0}$	$u_{4, 0} = 0$
t_1	1	$u_{0, 1} = 0$	$u_{1, 1}$	$u_{2, 1}$	$u_{3, 1}$	$u_{4, 1} = 0$
t_2	2	$u_{0, 2} = 0$	$u_{1, 2}$	$u_{2, 2}$	$u_{3, 2}$	$u_{4, 2} = 0$
t_3	3	$u_{0, 3} = 0$	$u_{1, 3}$	$u_{2, 3}$	$u_{3, 3}$	$u_{4, 3} = 0$
t_4	4	$u_{0, 4} = 0$	$u_{1, 4}$	$u_{2, 4}$	$u_{3, 4}$	$u_{4, 4} = 0$
t_5	5	$u_{0, 5} = 0$	$u_{1, 5}$	$u_{2, 5}$	$u_{3, 5}$	$u_{4, 5} = 0$

Consider the initial condition $u(x, 0) = x(4 - x)$

$\therefore \quad u_{1,0} = u(1, 0) = 3 \; ; \; u_{2,0} = 4 \; ; \; u_{3,0} = 3$

(First row in the table is completed)

Now we consider the relation

$$u_{i, j+1} = \frac{1}{2}\left[u_{i-1, j} + u_{i+1, j}\right] \qquad \ldots (1)$$

In particular, $u_{i, 1} = \frac{1}{2}\left[u_{i-1, 0} + u_{i+1, 0}\right]$

$$\therefore \quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+4) = 2$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(3+3) = 3$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(4+0) = 2$$

(Second row in the table is completed)

Again from (1), $u_{i, 2} = \frac{1}{2}\left[u_{i-1, 1} + u_{i+1, 1}\right]$

$$\therefore \quad u_{1,2} = \frac{1}{2}\left[u_{0,1} + u_{2,1}\right] = \frac{1}{2}(0+3) = 1.5$$

$$u_{2,2} = \frac{1}{2}\left[u_{1,1} + u_{3,1}\right] = \frac{1}{2}(2+2) = 2$$

$$u_{3,2} = \frac{1}{2}\left[u_{2,1} + u_{4,1}\right] = \frac{1}{2}(3+0) = 1.5$$

(Third row in the table is completed)

Also from (1), $u_{i, 3} = \frac{1}{2}\left[u_{i-1, 2} + u_{i+1, 2}\right]$

$$\therefore \quad u_{1,3} = \frac{1}{2}\left[u_{0,2} + u_{2,2}\right] = \frac{1}{2}(0+2) = 1$$

$$u_{2,3} = \frac{1}{2}\left[u_{1,2} + u_{3,2}\right] = \frac{1}{2}(1.5+1.5) = 1.5$$

$$u_{3,3} = \frac{1}{2}\left[u_{2,2} + u_{4,2}\right] = \frac{1}{2}(2+0) = 1$$

(Fourth row in the table is completed)

Also from (1), $u_{i,\,4} = \frac{1}{2}\left[u_{i-1,\,3} + u_{i+1,\,3}\right]$

$$\therefore \quad u_{1,\,4} = \frac{1}{2}\left[u_{0,\,3} + u_{2,\,3}\right] = \frac{1}{2}(0+1.5) = 0.75$$

$$u_{2,\,4} = \frac{1}{2}\left[u_{1,\,3} + u_{3,\,3}\right] = \frac{1}{2}(1+1) = 1$$

$$u_{3,\,4} = \frac{1}{2}\left[u_{2,\,3} + u_{4,\,3}\right] = \frac{1}{2}(1.5+0) = 0.75$$

(Fifth row in the table is completed)

Also from (1), $u_{i,\,5} = \frac{1}{2}\left[u_{i-1,\,4} + u_{i+1,\,4}\right]$

$$\therefore \quad u_{1,\,5} = \frac{1}{2}\left[u_{0,\,4} + u_{2,\,4}\right] = \frac{1}{2}(0+1) = 0.5$$

$$u_{2,\,5} = \frac{1}{2}\left[u_{1,\,4} + u_{3,\,4}\right] = \frac{1}{2}(0.75+0.75) = 0.75$$

$$u_{3,\,5} = \frac{1}{2}\left[u_{2,\,4} + u_{4,\,4}\right] = \frac{1}{2}(1+0) = 0.5$$

(Last row in the table is completed)

Thus the required values of $u_{i,\,j}$ are tabulated.

t \ x	0	1	2	3	4
0	0	**3**	4	**3**	0
1	0	**2**	**3**	**2**	0
2	0	**1.5**	**2**	**1.5**	0
3	0	**1**	**1.5**	**1**	0
4	0	**0.75**	**1**	**0.75**	0
5	0	**0.5**	**0.75**	**0.5**	0

9. *Solve* : $u_t = u_{xx}$ *subject to the conditions* $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \sin(\pi x)$ *for* $0 \le t \le 0.1$ *by taking* $h = 0.2$. *Write down the following values from the table*

(a) $u(0.2, 0.04)$ (b) $u(0.4, 0.08)$ (c) $u(0.6, 0.06)$

>> Comparing the standard form of the heat equation

$c^2 u_{xx} = u_t$ with the given equation $u_{xx} = u_t$ we have $c^2 = 1$

Since $h = 0.2$, $k = h^2/2c^2 = 0.04/2 = 0.02$

The values of x in $0 \leq x \leq 1$ with $h = 0.2$ are 0, 0.2, 0.4, 0.6, 0.8 and 1.
Also the values of t in the given range $0 \leq t \leq 0.1$ are 0, 0.02, 0.04, 0.06, 0.08 and 0.1

We have the following initial table, where in the values in the first and last column are zero since $u(0, t) = 0$ and $u(1, t) = 0$

t \ x		x_0	x_1	x_2	x_3	x_4	x_5
		0	0.2	0.4	0.6	0.8	1
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0} = 0$
t_1	0.02	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1} = 0$
t_2	0.04	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2} = 0$
t_3	0.06	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3} = 0$
t_4	0.08	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4} = 0$
t_5	0.1	$u_{0,5} = 0$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5}$	$u_{5,5} = 0$

Consider $u(x, 0) = \sin(\pi x)$

$$\therefore \quad u_{1,0} = u(0.2, 0) = \sin(\pi/5) = 0.59$$

$$u_{2,0} = u(0.4, 0) = \sin(2\pi/5) = 0.95$$

$$u_{3,0} = u(0.6, 0) = \sin(3\pi/5) = 0.95$$

$$u_{4,0} = u(0.8, 0) = \sin(4\pi/5) = 0.59$$

(First row in the table is completed)

Now we consider the relation

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right] \qquad \ldots (1)$$

Hence $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$$\therefore \quad u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0 + 0.95) = 0.475$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(0.59 + 0.95) = 0.77$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(0.95 + 0.59) = 0.77$$

$$u_{4,\,1} = \frac{1}{2}\left[u_{3,\,0} + u_{5,\,0}\right] = \frac{1}{2}\,(0.95 + 0) \quad = 0.475$$

(Second row in the table is completed)

Again from (1), $u_{i,\,2} = \frac{1}{2}\left[u_{i-1,\,1} + u_{i+1,\,1}\right]$

$$\therefore \quad u_{1,\,2} = \frac{1}{2}\left[u_{0,\,1} + u_{2,\,1}\right] = \frac{1}{2}\,(0 + 0.77) \quad = 0.385$$

$$u_{2,\,2} = \frac{1}{2}\left[u_{1,\,1} + u_{3,\,1}\right] = \frac{1}{2}\,(0.475 + 0.77) = 0.6225$$

$$u_{3,\,2} = \frac{1}{2}\left[u_{2,\,1} + u_{4,\,1}\right] = \frac{1}{2}\,(0.77 + 0.475) = 0.6225$$

$$u_{4,\,2} = \frac{1}{2}\left[u_{3,\,1} + u_{5,\,1}\right] = \frac{1}{2}\,(0.77 + 0) \quad = 0.385$$

(Third row in the table is completed)

Again from (1), $u_{i,\,3} = \frac{1}{2}\left[u_{i-1,\,2} + u_{i+1,\,2}\right]$

$$\therefore \quad u_{1,\,3} = \frac{1}{2}\left[u_{0,\,2} + u_{2,\,2}\right] = \frac{1}{2}\,(0 + 0.6225) \quad = 0.3113$$

$$u_{2,\,3} = \frac{1}{2}\left[u_{1,\,2} + u_{3,\,2}\right] = \frac{1}{2}\,(0.385 + 0.6225) = 0.504$$

$$u_{3,\,3} = \frac{1}{2}\left[u_{2,\,2} + u_{4,\,2}\right] = \frac{1}{2}\,(0.6225 + 0.385) = 0.504$$

$$u_{4,\,3} = \frac{1}{2}\left[u_{3,\,2} + u_{5,\,2}\right] = \frac{1}{2}\,(0.6225 + 0) \quad = 0.3113$$

(Fourth row in the table is completed)

Again from (1), $u_{i,\,4} = \frac{1}{2}\left[u_{i-1,\,3} + u_{i+1,\,3}\right]$

$$\therefore \quad u_{1,\,4} = \frac{1}{2}\left[u_{0,\,3} + u_{2,\,3}\right] = \frac{1}{2}\,(0 + 0.504) \quad = 0.252$$

$$u_{2,\,4} = \frac{1}{2}\left[u_{1,\,3} + u_{3,\,3}\right] = \frac{1}{2}\,(0.3113 + 0.504) = 0.408$$

$$u_{3,\,4} = \frac{1}{2}\left[u_{2,\,3} + u_{4,\,3}\right] = \frac{1}{2}\,(0.504 + 0.3113) = 0.408$$

$$u_{4,\,4} = \frac{1}{2}\left[u_{3,\,3} + u_{5,\,3}\right] = \frac{1}{2}\,(0.504 + 0) \quad = 0.252$$

(Fifth row in the table is completed)

Again from (1), $u_{i,\,5} = \frac{1}{2}\left[u_{i-1,\,4} + u_{i+1,\,4}\right]$

$\therefore \quad u_{1,\,5} = \frac{1}{2}\left[u_{0,\,4} + u_{2,\,4}\right] = \frac{1}{2}(0 + 0.408) = 0.204$

$u_{2,\,5} = \frac{1}{2}\left[u_{1,\,4} + u_{3,\,4}\right] = \frac{1}{2}(0.252 + 0.408) = 0.33$

$u_{3,\,5} = \frac{1}{2}\left[u_{2,\,4} + u_{4,\,4}\right] = \frac{1}{2}(0.408 + 0.252) = 0.33$

$u_{4,\,5} = \frac{1}{2}\left[u_{3,\,4} + u_{5,\,4}\right] = \frac{1}{2}(0.408 + 0) = 0.204$

(Last row in the table is completed)

Thus the required values of $u_{i,\,j}$ are tabulated.

t \ x	0	0.2	0.4	0.6	0.8	1
0	0	**0.59**	**0.95**	**0.95**	**0.59**	0
0.02	0	**0.475**	**0.77**	**0.77**	**0.475**	0
0.04	0	**0.385**	**0.6225**	**0.6225**	**0.385**	0
0.06	0	**0.3113**	**0.504**	**0.504**	**0.3113**	0
0.08	0	**0.252**	**0.408**	**0.408**	**0.252**	0
0.1	0	**0.204**	**0.33**	**0.33**	**0.204**	0

Also from the table we have the following values as required.

(a) $u(0.2,\ 0.04) = \mathbf{0.385}$ (b) $u(0.4,\ 0.08) = \mathbf{0.408}$ (c) $u(0.6,\ 0.06) = \mathbf{0.504}$

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10. *Solve : $u_{xx} = 32\,u_t$ subject to the conditions $u(0,\ t) = 0$, $u(1,\ t) = t$ and $u(x,\ 0) = 0$. Find the values of u upto $t = 5$ by Schmidt's process taking $h = 1/4$. Also extract the following values :*

(*a*) $u(0.75,\ 4)$ (*b*) $u(0.5,\ 5)$ (*c*) $u(0.25,\ 4)$

>> Comparing the standard form of the heat equation $c^2 u_{xx} = u_t$ with the given equation $u_{xx} = 32\,u_t$ we have $c^2 = 1/32$

Since $h = 1/4$, $k = h^2/2c^2 = 1$. The values of x in $0 \le x \le 1$ with $h = 1/4$ are $0,\ 0.25,\ 0.5,\ 0.75,\ 1$ and the values of t with $k = 1$ are $0,\ 1,\ 2,\ 3,\ 4,\ 5$

We have the following initial table wherein the values in the first coloumn and first row are zero by the given initial conditions $u(0,\ t) = 0$ and $u(x,\ 0) = 0$

$t \backslash x$		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0$	$u_{2,0} = 0$	$u_{3,0} = 0$	$u_{4,0} = 0$
t_1	1	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$
t_2	2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$
t_3	3	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$
t_4	4	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$
t_5	5	$u_{0,5} = 0$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5}$

Now we consider the relation

$$u_{i,\,j+1} = \frac{1}{2}\left[u_{i-1,\,j} + u_{i+1,\,j}\right]$$

Hence $u_{i,\,1} = \frac{1}{2}\left[u_{i-1,\,0} + u_{i+1,\,0}\right]$

$\therefore \quad u_{1,\,1} = \frac{1}{2}\left[u_{0,\,0} + u_{2,\,0}\right] = 0 \;;\; u_{2,\,1} = \frac{1}{2}\left[u_{1,\,0} + u_{3,\,0}\right] = 0$

$\therefore \quad u_{3,\,1} = \frac{1}{2}\left[u_{2,\,0} + u_{4,\,0}\right] = 0 \;;\; u_{4,\,1} = \frac{1}{2}\left[u_{3,\,0} + u_{5,\,0}\right]$ becomes in applicable.

$\therefore \quad u(4,\,1) = u(x_4,\,t_1) = t_1 = 1.$

(Second row in the table is completed)

Again from (1) $u_{i,\,2} = \frac{1}{2}\left[u_{i-1,\,1} + u_{i+1,\,1}\right]$

$\therefore \quad u_{1,\,2} = \frac{1}{2}\left[u_{0,\,1} + u_{2,\,1}\right] = 0 \;;\; u_{2,\,2} = \frac{1}{2}\left[u_{1,\,1} + u_{3,\,1}\right] = 0$

$u_{3,\,2} = \frac{1}{2}\left[u_{2,\,1} + u_{4,\,1}\right] = 0.5 \;;\; u_{4,\,2} = u(x_4,\,t_2) = t_2 = 2$

(Third row in the table is completed)

Again from (1) $u_{i,\,3} = \frac{1}{2}\left[u_{i-1,\,2} + u_{i+1,\,2}\right]$

$\therefore \quad u_{1,\,3} = \frac{1}{2}\left[u_{0,\,2} + u_{2,\,2}\right] = 0 \;;\; u_{2,\,3} = \frac{1}{2}\left[u_{1,\,2} + u_{3,\,2}\right] = 0.25$

$u_{3,\,3} = \frac{1}{2}\left[u_{2,\,2} + u_{4,\,2}\right] = 1 \;;\; u_{4,\,3} = u(x_4,\,t_3) = t_3 = 3$

(Fourth row in the table is completed)

Again from (1) $u_{i,4} = \frac{1}{2}\left[u_{i-1,3} + u_{i+1,3}\right]$

$\therefore \quad u_{1,4} = \frac{1}{2}\left[u_{0,3} + u_{2,3}\right] = 0.125 \; ; \; u_{2,4} = \frac{1}{2}\left[u_{1,3} + u_{3,3}\right] = 0.5$

$u_{3,4} = \frac{1}{2}\left[u_{2,3} + u_{4,3}\right] = 1.625 \; ; \; u_{4,4} = u(x_4, t_4) = t_4 = 4$

(*Fifth row in the table is completed*)

Lastly from (1) $u_{i,5} = \frac{1}{2}\left[u_{i-1,4} + u_{i+1,4}\right]$

$\therefore \quad u_{1,5} = \frac{1}{2}\left[u_{0,4} + u_{2,4}\right] = 0.25 \; ; u_{2,5} = \frac{1}{2}\left[u_{1,4} + u_{3,4}\right] = 0.875$

$u_{3,5} = \frac{1}{2}\left[u_{2,4} + u_{4,4}\right] = 2.25 \; ; \; u_{4,5} = u(x_4, t_5) = t_5 = 5$

(*Last row in the table is completed*)

Thus the required values of $u_{i,j}$ are tabulated.

t \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	**0**	**0**	**0**	**1**
2	0	**0**	**0**	**0.5**	**2**
3	0	**0**	**0.25**	**1**	**3**
4	0	**0.125**	**0.5**	**1.625**	**4**
5	0	**0.25**	**0.875**	**2.25**	**5**

Also we have from the table :

(a) $u(0.75, 4) = 1.625$ (b) $u(0.5, 5) = 0.875$

(c) $u(0.25, 4) = 0.125$

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11. *Solve the equation* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 1$ *at* $t = 0.002$ *given that* $h = 0.1$, $k = 0.001$ *and the conditions* $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = f(x)$ *where* $f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases}$

>> Comparing $c^2 u_{xx} = u_t$ with the given equation $u_{xx} = u_t$, we have $c^2 = 1$.
[*It should be noted that* k *is specifically given to be* 0.001 *and we cannot use Bendre - Schmidt formula which is a deduction from the Schmidt formula choosing* $kc^2/h^2 = 1/2$. *Here* $kc^2/h^2 = 0.1$]

We have to use the Schmidt explicit formula in the form

$$u_{i,\,j+1} = a\,u_{i-1,\,j} + (1-2a)\,u_{i,\,j} + a\,u_{i+1,\,j} \qquad \ldots (1)$$

where $a = kc^2/h^2$ which is equal to 0.1. Hence (1) becomes

$$u_{i,\,j+1} = 0.1\,u_{i-1,\,j} + 0.8\,u_{i,\,j} + 0.1\,u_{i+1,\,j} \qquad \ldots (2)$$

The values of x in $0 \le x \le 1$ with $h = 0.1$ are 0, 0.1, 0.2, ··· 1.0 and the values of t are 0, 0.001, 0.002 since $k = 0.001$. We have the following initial table wherein the values in the first and last column are zero since $u(0, t) = 0$, $u(1, t) = 0$

t \ x		x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
t_0	0	$u_{0,0}=0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0}$	$u_{6,0}$	$u_{7,0}$	$u_{8,0}$	$u_{9,0}$	$u_{10,0}=0$
t_1	0.001	$u_{0,1}=0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1}$	$u_{6,1}$	$u_{7,1}$	$u_{8,1}$	$u_{9,1}$	$u_{10,1}=0$
t_2	0.002	$u_{0,2}=0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2}$	$u_{6,2}$	$u_{7,2}$	$u_{8,2}$	$u_{9,2}$	$u_{10,2}=0$

Let us consider $u(x, 0) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases}$

$\therefore \quad u_{1,0} = u(x_1, 0) = u(0.1, 0) = 2 \times 0.1 = 0.2$

$u_{2,0} = u(0.2, 0) = 2 \times 0.2 = 0.4 \;;\; u_{3,0} = 2 \times 0.3 = 0.6$

$u_{4,0} = 2 \times 0.4 = 0.8 \;;\; u_{5,0} = 2 \times 0.5 = 1 \;;$

$u_{6,0} = 2(1-0.6) = 0.8 \;;\; u_{7,0} = 2(1-0.7) = 0.6 \;;$

$u_{8,0} = 2(1-0.8) = 0.4 \;;\; u_{9,0} = 2(1-0.9) = 0.2$

Now from (2), we have when $j = 0$

$$u_{i,\,1} = 0.1\,u_{i-1,\,0} + 0.8\,u_{i,\,0} + 0.1\,u_{i+1,\,0}$$

$\therefore \quad u_{1,1} = 0.1\,u_{0,0} + 0.8\,u_{1,0} + 0.1\,u_{2,0} = 0.2$

$u_{2,1} = 0.1\,u_{1,0} + 0.8\,u_{2,0} + 0.1\,u_{3,0} = 0.4$

$$u_{3,1} = 0.1\,u_{2,0} + 0.8\,u_{3,0} + 0.1\,u_{4,0} = 0.6$$

$$u_{4,1} = 0.1\,u_{3,0} + 0.8\,u_{4,0} + 0.1\,u_{5,0} = 0.8$$

$$u_{5,1} = 0.1\,u_{4,0} + 0.8\,u_{5,0} + 0.1\,u_{6,0} = 0.96$$

$$u_{6,1} = 0.1\,u_{5,0} + 0.8\,u_{6,0} + 0.1\,u_{7,0} = 0.8$$

Similarly we can obtain $u_{7,1} = 0.6$; $u_{8,1} = 0.4$; $u_{9,1} = 0.2$

Again from (2) we have when $j = 1$,

$$u_{i,2} = 0.1\,u_{i-1,1} + 0.8\,u_{i,1} + 0.1\,u_{i+1,1}$$

$$\therefore \quad u_{1,2} = 0.1\,u_{0,1} + 0.8\,u_{1,1} + 0.1\,u_{2,1} = 0.2$$

$$u_{2,2} = 0.1\,u_{1,1} + 0.8\,u_{2,1} + 0.1\,u_{3,1} = 0.4$$

$$u_{3,2} = 0.1\,u_{2,1} + 0.8\,u_{3,1} + 0.1\,u_{4,1} = 0.6$$

$$u_{4,2} = 0.1\,u_{3,1} + 0.8\,u_{4,1} + 0.1\,u_{5,1} = 0.796$$

$$u_{5,2} = 0.1\,u_{4,1} + 0.8\,u_{5,1} + 0.1\,u_{6,1} = 0.928$$

Similarly we can obtain $u_{6,2} = 0.796$; $u_{7,2} = 0.6$; $u_{8,2} = 0.4$; $u_{9,2} = 0.2$

Thus the required values of $u_{i,j}$ are tabulated

$t \backslash x$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**0.8**	**0.6**	**0.4**	**0.2**	0
0.001	0	**0.2**	**0.4**	**0.6**	**0.8**	**0.96**	**0.8**	**0.6**	**0.4**	**0.2**	0
0.002	0	**0.2**	**0.4**	**0.6**	**0.796**	**0.928**	**0.796**	**0.6**	**0.4**	**0.2**	0

---

12. *Solve numerically the equation* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ *subject to the conditions* $u(0, t) = 0 = u(1, t)$, $t \geq 0$ *and* $u(x, 0) = \sin \pi x, 0 \leq x \leq 1$.
Carryout computations for two levels taking $h = 1/3$ and $k = 1/36$

\>> We have Schmidt explicit formula,

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j} \quad \ldots (1)$$

where $\alpha = k c^2/h^2$. We have $c^2 = 1$, $h = 1/3$, $k = 1/36$ $\therefore$ $\alpha = 1/4$

Hence (1) becomes

$$u_{i,j+1} = \frac{1}{4}\, u_{i-1,j} + \frac{1}{2}\, u_{i,j} + \frac{1}{4}\, u_{i+1,j}$$

ie., $$u_{i,\, j+1} = \frac{1}{4}\left[u_{i-1,\, j} + 2\, u_{i,\, j} + u_{i+1,\, j} \right] \qquad \ldots (2)$$

Since $h = 1/3$, the values of x in $0 \le x \le 1$ are $x_0 = 0, x_1 = 1/3, x_2 = 2/3$ & $x_3 = 1$.

$$u(0, t) = 0 \Rightarrow u_{0,\,0} = 0 = u_{0,\,1} = u_{0,\,2} = \cdots$$

$$u(1, t) = 0 \Rightarrow u_{3,\,0} = 0 = u_{3,\,1} = u_{3,\,2} = \cdots$$

Also, $u(x, 0) = \sin \pi x$ and hence we have

$$u(x_1, 0) = u_{1,\,0} = \sin(\pi/3) = 0.866$$

$$u(x_2, 0) = u_{2,\,0} = \sin(2\pi/3) = 0.866$$

We shall compute $u_{1,\,1}$, $u_{2,\,1}$ (*first level*) and $u_{1,\,2}$, $u_{2,\,2}$ (*second level*) by using (2).

Thus we have,

$$u_{1,\,1} = \frac{1}{4}\left[u_{0,\,0} + 2\, u_{1,\,0} + u_{2,\,0} \right] = \mathbf{0.6495}$$

$$u_{2,\,1} = \frac{1}{4}\left[u_{1,\,0} + 2\, u_{2,\,0} + u_{3,\,0} \right] = \mathbf{0.6495}$$

$$u_{1,\,2} = \frac{1}{4}\left[u_{0,\,1} + 2\, u_{1,\,1} + u_{2,\,1} \right] = \mathbf{0.487125}$$

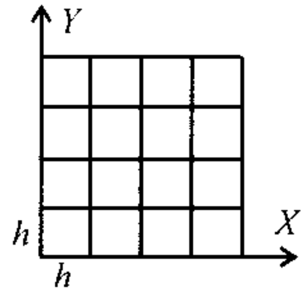
$$u_{2,\,2} = \frac{1}{4}\left[u_{1,\,1} + 2\, u_{2,\,1} + u_{3,\,1} \right] = \mathbf{0.487125}$$

7.43 Numerical solution of the Laplace's equation in two dimensions

Laplace's equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots (1)$$

We consider a rectangular region R for which $u(x, y)$ is known at the boundary. Let us suppose that the region is such that it can be divided into a network of square mesh of side h.



We shall substitute the finite difference approximation for the partial derivatives present in (1) [F_7 *and* F_8 *are to be used*].

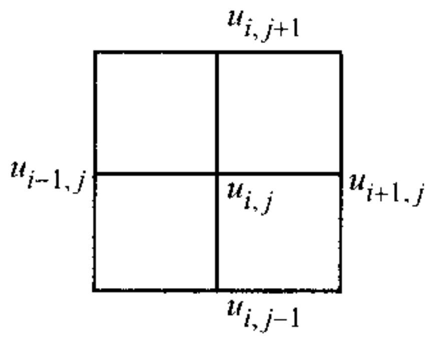
Hence we have

$$\frac{1}{h^2}\left[u_{i-1,\,j}-2\,u_{i,\,j}+u_{i+1,\,j}\right]+\frac{1}{h^2}\left[u_{i,\,j-1}-2\,u_{i,\,j}+u_{i,\,j+1}\right]=0$$

That is, $u_{i-1,\,j}+u_{i+1,\,j}+u_{i,\,j-1}+u_{i,\,j+1}=4\,u_{i,\,j}$

or $$u_{i,\,j}=\frac{1}{4}\left[u_{i-1,\,j}+u_{i+1,\,j}+u_{i,\,j+1}+u_{i,\,j-1}\right] \qquad \ldots(2)$$

This is called the **standard five point formula**. It may be observed that the value of $u_{i,\,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right , above & below as exhibited in the following figure.

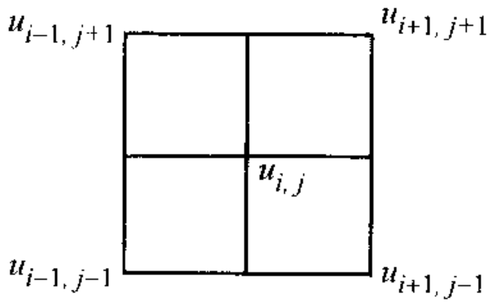


Note : 1. *Since the Laplace equation remains invariant when the co-ordinates are rotated through an angle of* 45° *we can also have the formula in the form*

$$u_{i,\,j}=\frac{1}{4}\left[u_{i-1,\,j+1}+u_{i+1,\,j-1}+u_{i+1,\,j+1}+u_{i-1,\,j-1}\right] \qquad \ldots(3)$$

It may be observed that the value of $u_{i,\,j}$ *is the average of its values at the four neighbouring diagonal mesh points. Accordingly (3) is called* ***diagonal five point formula.***

[*Figure in the following page*]



2. *The accuracy of the values of $u_{i,\,j}$ can be improved by successive application of the formula (2). We regard the obtained values of $u_{i,\,j}$ as the initial approximation denoted by $u_{i,\,j}^{(0)}$. Then the first iteration is carried out by using the latest iterative value available on hand where we have,*

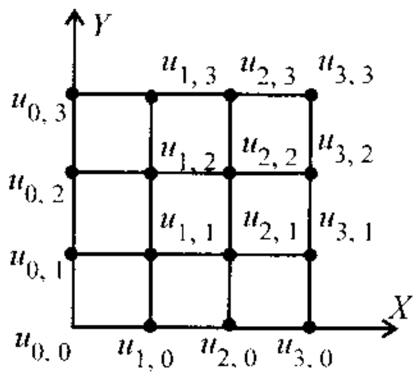
$$u_{i,\,j}^{(1)} = \frac{1}{4}\left[u_{i-1,\,j}^{(0)} + u_{i+1,\,j}^{(0)} + u_{i,\,j+1}^{(0)} + u_{i,\,j-1}^{(0)}\right]$$

The procedure will be repeated till we get the values to the desired degree of accuracy. This process is called ***Liebmann's iteration process***

Working procedure for problems

Type - 1 : ***Odd number of squares are being formed.***

(Generally 9 squares in a practical problem)



We need to compute $u_{1,\,1}$; $u_{2,\,1}$; $u_{2,\,2}$; $u_{1,\,2}$

After establishing the standard five point formula we have said that the value of $u_{i,\,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below.

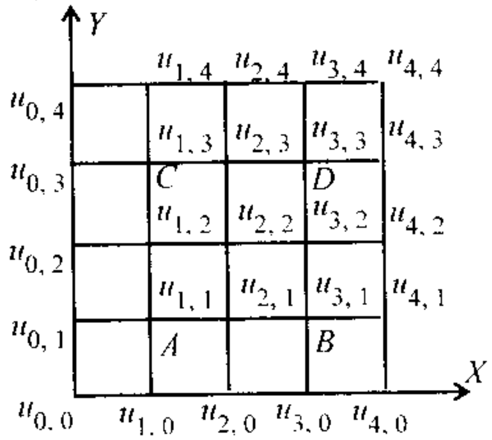
$$\therefore \quad u_{1,\,1} = \frac{1}{4}\left[u_{0,\,1} + u_{2,\,1} + u_{1,\,2} + u_{1,\,0}\right]$$

$$u_{2,\,1} = \frac{1}{4}\left[u_{1,\,1} + u_{3,\,1} + u_{2,\,2} + u_{2,\,0}\right]$$ and two similar results.

We substitute the given values to obtain a system of equations in the unknowns and solve to get the desired values.

Type - 2 : ***Even number of squares are being formed***

(Generally 16 squares in a practical problem)



We note that $u_{2,\,2}$ is exactly at the centre of the region and **we first compute this value** by the standard five point formula.

$$u_{2,\,2} = \frac{1}{4}\left[u_{0,\,2} + u_{4,\,2} + u_{2,\,4} + u_{2,\,0}\right]$$

Next we note that the entire square region R is the union of four square blocks and we focus on these square blocks. The mesh points $A\,(u_{1,\,1})$; $B\,(u_{3,\,1})$; $C\,(u_{1,\,3})$ and $D\,(u_{3,\,3})$ are at the centre of these four blocks. Hence we compute these values by the diagonal five point formula which says that the value of $u_{i,\,j}$ is the average of its values at the four neighbouring diagonal mesh points.

$$\therefore \qquad u_{1,\,1} = \frac{1}{4}\left[u_{0,\,0} + u_{2,\,2} + u_{0,\,2} + u_{2,\,0}\right]$$

$$u_{3,\,1} = \frac{1}{4}\left[u_{2,\,0} + u_{4,\,2} + u_{2,\,2} + u_{4,\,0}\right]$$ and two similar results.

Lastly we need to compute $u_{1,\,2}$; $u_{2,\,1}$; $u_{2,\,3}$; $u_{3,\,2}$

These can be found by the standard five point formula.

$$u_{1,\,2} = \frac{1}{4}\left[u_{0,\,2} + u_{2,\,2} + u_{1,\,3} + u_{1,\,1}\right]$$

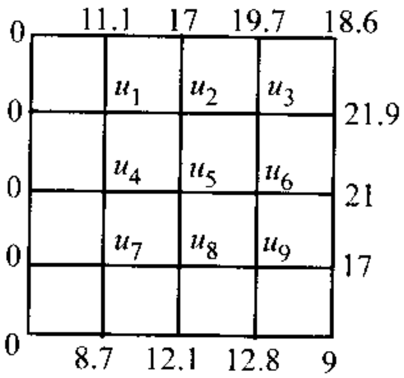
$$u_{2,\,1} = \frac{1}{4}\left[u_{1,\,1} + u_{3,\,1} + u_{2,\,2} + u_{2,\,0}\right]$$ and two similar results.

Liebmann's iterative process can be carried out for better accuracy in the values obtained.

Note : *We use the abbreviation* **S.F** *for the standard five point formula and* **D.F** *for the diagonal five point formula*

WORKED PROBLEMS

13. *Solve Laplace's equation* $u_{xx} + u_{yy} = 0$ *for the following square mesh with boundary values as shown in the following figure.*



>> u_5 is located at the centre of the region.

$$\therefore \quad u_5 = \frac{1}{4}(0 + 21 + 17 + 12.1) = 12.525 \text{ by S.F}$$

Next we shall compute u_7, u_9, u_1, u_3 by applying D.F

$$\therefore \quad u_7 = \frac{1}{4}(0 + 12.525 + 0 + 12.1) = 6.15625$$

$$u_9 = \frac{1}{4}(12.1 + 21 + 12.525 + 9) = 13.65625$$

$$u_1 = \frac{1}{4}(0 + 17 + 0 + 12.525) = 7.38125$$

$$u_3 = \frac{1}{4}(12.525 + 18.6 + 17 + 21) = 17.28125$$

Finally we shall compute u_2, u_4, u_6 and u_8 by S.F.

$$u_2 = \frac{1}{4}(7.38125 + 17.28125 + 17 + 12.525) = 13.546875$$

$$u_4 = \frac{1}{4}(0 + 12.525 + 7.38125 + 6.15625) = 6.515625$$

$$u_6 = \frac{1}{4}\,(12.525 + 21 + 17.28125 + 13.65625) = 16.115625$$

$$u_8 = \frac{1}{4}\,(6.15625 + 13.65625 + 12.525 + 12.1) = 11.109375$$

Thus the required values of $u(x, y)$ at the interior mesh points correct to two decimal places are as follows.

$$u_1 = 7.38 \quad u_2 = 13.55 \quad u_3 = 17.28 \quad u_4 = 6.52 \quad u_5 = 12.53$$

$$u_6 = 16.12 \quad u_7 = 6.16 \quad u_8 = 11.11 \quad u_9 = 13.66$$

---

14. *Solve $\nabla^2 u = 0$ in the square region bounded by the co ordinate axes and the lines $x = 4,\ y = 4$ with the boundary conditions given by the analytical expressions,*

(i) $u(0, y) = 0$ *for* $0 \le y \le 4$

(ii) $u(4, y) = 12 + y$ *for* $0 \le y \le 4$

(iii) $u(x, 0) = 3x$ *for* $0 \le x \le 4$

(iv) $u(x, 4) = x^2$ *for* $0 \le x \le 4$

Also employ Liebmann's iteration process to compute the second iterative values of $u(x, y)$ correct to two decimal places.

>> We have $\nabla^2 u = 0$ represented by $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in two dimensions.

We shall divide the square region into 16 squares of side one unit.

We shall derive the values of $u(x, y)$ on the boundary from the given expressions.

(i) $u(0, y) = 0 \Rightarrow u(0, 1) = 0 = u(0, 2) = u(0, 3) = u(0, 4)$

(ii) $u(4, y) = 12 + y \Rightarrow u(4, 0) = 12\ ;\ u(4, 1) = 13\ ;$

$u(4, 2) = 14\ ;\ u(4, 3) = 15\ ;\ u(4, 4) = 16$

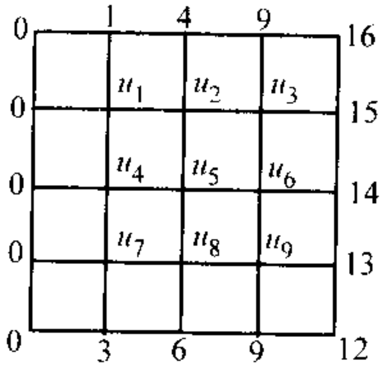
(iii) $u(x, 0) = 3x \Rightarrow u(0, 0) = 0\ ;\ u(1, 0) = 3\ ;$

$u(2, 0) = 6\ ;\ u(3, 0) = 9\ ;\ u(4, 0) = 12$

(iv) $u(x, 4) = x^2 \Rightarrow u(0, 4) = 0\ ;\ u(1, 4) = 1\ ;$

$u(2, 4) = 4\ ;\ u(3, 4) = 9\ ;\ u(4, 4) = 16.$

We shall represent these values on the square region and let $u_1, u_2, \cdots u_9$ be the interior mesh points of the region.



u_5 is located at the centre of the region.

$\therefore \quad u_5 = \frac{1}{4}(0+14+4+6) = 6$ by applying S.F

Next we apply D.F to compute u_7, u_9, u_1, u_3

$$u_7 = \frac{1}{4}(0+6+0+6) = 3 \quad ; \quad u_9 = \frac{1}{4}(6+14+6+12) = 9.5$$

$$u_1 = \frac{1}{4}(0+4+0+6) = 2.5 \quad ; \quad u_3 = \frac{1}{4}(6+16+4+14) = 10$$

Now we shall compute u_2, u_4, u_6, u_8 by S.F

$$u_2 = \frac{1}{4}(2.5+10+4+6) = 5.625$$

$$u_4 = \frac{1}{4}(0+6+2.5+3) = 2.875$$

$$u_6 = \frac{1}{4}(6+14+10+9.5) = 9.875$$

$$u_8 = \frac{1}{4}(3+9.5+6+6) = 6.125$$

These values are regarded as the initial approximations to commence the Liebmann's iterative process for greater accuracy. We compute u_i ($i = 1$ to 9) in the serial order by using the latest iterative value on hand by applying the standard five point formula only.

First iteration

$$u_1 = \frac{1}{4}(0+5.625+1+2.875) = 2.375$$

$$u_2 = \frac{1}{4}(2.375+10+4+6) = 5.59375$$

$$u_3 = \frac{1}{4}(5.59375+15+9+9.875) = 9.8671875$$

$$u_4 = \frac{1}{4}\,(0 + 6 + 2.375 + 3) = 2.84375$$

$$u_5 = \frac{1}{4}\,(2.84375 + 9.875 + 5.59375 + 6.125) = 6.109375$$

$$u_6 = \frac{1}{4}\,(6.109375 + 14 + 9.8671875 + 9.5) = 9.8691406$$

$$u_7 = \frac{1}{4}\,(0 + 6.125 + 2.84375 + 3) = 2.9921875$$

$$u_8 = \frac{1}{4}\,(2.9921875 + 9.5 + 6.109375 + 6) = 6.1503906$$

$$u_9 = \frac{1}{4}\,(6.1503906 + 13 + 9.8691406 + 9) = 9.5048828$$

Now we have the first iterative values correct to three decimal places

$u_1 = 2.375 \quad u_2 = 5.594 \quad u_3 = 9.867 \quad u_4 = 2.844 \quad u_5 = 6.109$

$u_6 = 9.869 \quad u_7 = 2.992 \quad u_8 = 6.150 \quad u_9 = 9.505$

Second iteration

$$u_1 = \frac{1}{4}\,(0 + 5.594 + 1 + 2.844) \approx 2.36$$

$$u_2 = \frac{1}{4}\,(2.36 + 9.867 + 4 + 6.109) = 5.584$$

$$u_3 = \frac{1}{4}\,(5.584 + 15 + 9 + 9.869) = 9.863$$

$$u_4 = \frac{1}{4}\,(0 + 6.109 + 2.36 + 2.992) = 2.865$$

$$u_5 = \frac{1}{4}\,(2.865 + 9.869 + 5.584 + 6.150) = 6.117$$

$$u_6 = \frac{1}{4}\,(6.117 + 14 + 9.863 + 9.505) = 9.871$$

$$u_7 = \frac{1}{4}\,(0 + 6.150 + 2.865 + 3) = 3.004$$

$$u_8 = \frac{1}{4}\,(3.004 + 9.505 + 6.117 + 6) = 6.156$$

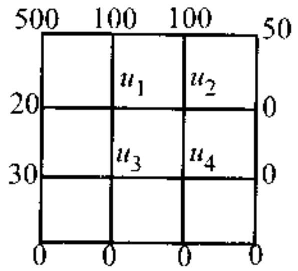
$$u_9 = \frac{1}{4}\,(6.156 + 13 + 9.871 + 9) = 9.507$$

Thus the **required second iterative values** correct to two decimal places are as follows.

$\mathbf{u_1 = 2.36 \quad u_2 = 5.58 \quad u_3 = 9.86 \quad u_4 = 2.87 \quad u_5 = 6.12}$

$\mathbf{u_6 = 9.87 \quad u_7 = 3 \quad u_8 = 6.16 \quad u_9 = 9.51}$

15. *Solve :* $u_{xx} + u_{yy} = 0$ *in the following square region with the boundary conditions as indicated in the figure.*



>> We shall apply standard five point formula for u_1, u_2, u_3, u_4 to obtain a system of equations.

$$u_1 = \frac{1}{4}(20 + u_2 + 100 + u_3) \; ; \; u_2 = \frac{1}{4}(u_1 + 0 + 100 + u_4)$$

$$u_3 = \frac{1}{4}(30 + u_4 + u_1 + 0) \quad ; \; u_4 = \frac{1}{4}(u_3 + 0 + u_2 + 0)$$

Now we have a system of equations to be solved.

$$4u_1 - u_2 - u_3 = 120 \quad \ldots (1)$$

$$-u_1 + 4u_2 - u_4 = 100 \quad \ldots (2)$$

$$-u_1 + 4u_3 - u_4 = 30 \quad \ldots (3)$$

$$-u_2 - u_3 + 4u_4 = 0 \quad \ldots (4)$$

Let us eliminate u_1 from (1) and (2) ; (2) and (3)

That is $15u_2 - u_3 - 4u_4 = 520$ ... (5)

$$4u_2 - 4u_3 = 70 \quad \ldots (6)$$

We shall now eliminate u_4 from (4) and (5)

That is $14u_2 - 2u_3 = 520$ ... (7)

Let us solve (6) and (7) : $2u_2 - 2u_3 = 35$ ... (6)

$$14u_2 - 2u_3 = 520 \quad \ldots (7)$$

$\therefore \quad u_2 = 40.4, \; u_3 = 22.9$

Hence from (1) $u_1 = 45.825$ and from (4) $u_4 = 15.825$

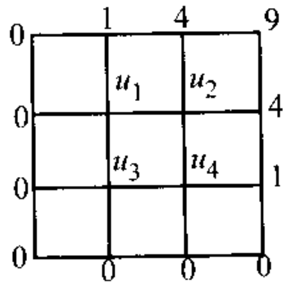
Thus the required values at the interior mesh points are

$$\mathbf{u_1 = 45.825,\ u_2 = 40.4,\ u_3 = 22.9,\ u_4 = 15.825}$$

Note : *We can employ iterative process for better accuracy in the values.*

---

16. *Evaluate the function* $u(x, y)$ *satisfying* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *at the pivotal points given the values on the boundary as indicated in the figure*



>> Let us apply the standard five point formula for u_1, u_2, u_3, u_4

$$u_1 = \frac{1}{4}(0 + u_2 + 1 + u_3)\ ;\ u_2 = \frac{1}{4}(u_1 + 4 + 4 + u_4)$$

$$u_3 = \frac{1}{4}(0 + u_4 + u_1 + 0)\ ;\ u_4 = \frac{1}{4}(u_3 + 1 + u_2 + 0)$$

Now we have a system of equations

$$4u_1 - u_2 - u_3 = 1 \quad \ldots (1)$$

$$-u_1 + 4u_2 - u_4 = 8 \quad \ldots (2)$$

$$-u_1 + 4u_3 - u_4 = 0 \quad \ldots (3)$$

$$-u_2 - u_3 + 4u_4 = 1 \quad \ldots (4)$$

Eliminating u_1 from (1) and (2) ; (2) and (3) we have

$$15u_2 - u_3 - 4u_4 = 33 \quad \ldots (5)$$

$$4u_2 - 4u_3 = 8 \text{ or } u_2 - u_3 = 2 \quad \ldots (6)$$

Eliminating u_4 from (4) and (5) we have

$$14u_2 - 2u_3 = 34 \quad \ldots (7)$$

Solving (6) and (7) we get $u_3 = 0.5,\ u_2 = 2.5$

From (1) we get $u_1 = 1$ and from (4) we get $u_4 = 1$

Thus the required values at the interior mesh points are

$$\mathbf{u_1 = 1,\ u_2 = 2.5,\ u_3 = 0.5,\ u_4 = 1}$$

---

17. *Solve the Laplace's equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *for* $0 < x < 1, 0 < y < 1$, *given that* $u(x, 0) = u(0, y) = 0$, $u(x, 1) = 6x$, $0 < x \le 1$ *and* $u(1, y) = 3y$, $0 < y < 1$ *Divide the region into 9 square meshes.*

>> Since the region is divided into 9 square meshes the stepsize for both x and y is 1/3. The points of division are 1/3 and 2/3. The values of x and y are 0, 1/3, 2/3, 1. The values of $u(x, y)$ on the boundary are found by using the given data.

(i) $u(x, 0) = 0 \Rightarrow u(0, 0) = 0, \; u(1/3, 0) = 0$

$u(2/3, 0) = 0, \; u(1, 0) = 0$

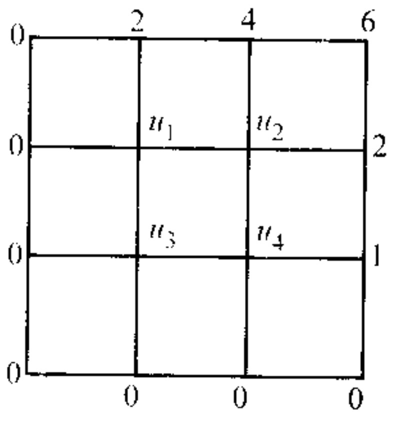
(ii) $u(0, y) = 0 \Rightarrow u(0, 0) = 0, \; u(0, 1/3) = 0$

$u(0, 2/3) = 0, \; u(0, 1) = 0$

(iii) $u(x, 1) = 6x \Rightarrow u(1/3, 1) = 2, \; u(2/3, 1) = 4 \quad u(1, 1) = 6$

(iv) $u(1, y) = 3y \Rightarrow u(1, 1/3) = 1, \; u(1, 2/3) = 2$

The values are represented in the following figure.



Let u_1, u_2, u_3, u_4 be the interior mesh points of the region. We shall apply standard five point formula for u_1, u_2, u_3, u_4 that leads to the following equations.

$$u_1 = \frac{1}{4}(0 + u_2 + 2 + u_3) \; ; \quad u_2 = \frac{1}{4}(u_1 + 2 + u_4 + 4)$$

$$u_3 = \frac{1}{4}(0 + u_4 + 0 + u_1) \; ; \quad u_4 = \frac{1}{4}(u_3 + 1 + 0 + u_2)$$

Thus we have to solve the following system of equations :

$$4u_1 - u_2 - u_3 = 2 \qquad \ldots (1)$$

$$-u_1 + 4u_2 - u_4 = 6 \qquad \ldots (2)$$

$$-u_1 + 4u_3 - u_4 = 0 \qquad \ldots (3)$$

$$-u_2 - u_3 + 4u_4 = 1 \qquad \ldots (4)$$

Eliminating u_1 from (1), (2) and (2), (3) we have

$$15u_2 - u_3 - 4u_4 = 26 \qquad \ldots (5)$$

$$4u_2 - 4u_3 = 6 \qquad \ldots (6)$$

Also (4) + (5) will give us

$$14u_2 - 2u_3 = 27 \qquad \ldots (7)$$

From (6) and (7) we get $u_2 = 2$, $u_3 = 1/2$

Further we obtain from (1), (2) $u_1 = 9/8$, $u_4 = 7/8$

Thus the required $\mathbf{u_1 = 9/8,\ u_2 = 2,\ u_3 = 1/2,\ u_4 = 7/8}$

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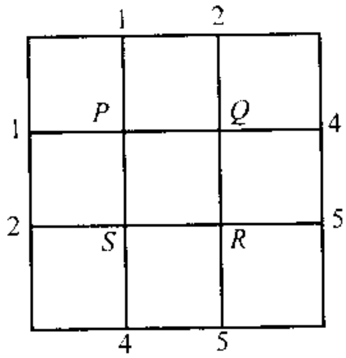
18. *Solve the elliptic partial differential equation for the following square mesh using the 5 point difference formula by setting up the linear equations at the unknown points P, Q, R, S*



>> The linear equations for the unknowns by applying the 5 point difference formula are as follows:

$$P = \frac{1}{4}(1 + Q + 1 + S) \ ; \quad Q = \frac{1}{4}(P + 4 + R + 2)$$

$$R = \frac{1}{4}(S + 5 + 5 + Q) \ ; \quad S = \frac{1}{4}(2 + R + 4 + P)$$

That is,

$$4P - Q - S = 2 \qquad \ldots (1)$$

$$-P + 4Q - R = 6 \qquad \ldots (2)$$

$$-Q + 4R - S = 10 \qquad \ldots (3)$$

$$-P - R + 4S = 6 \qquad \ldots (4)$$

Thus by solving we get $\mathbf{P = 2,\ Q = 3,\ S = 3,\ R = 4}$

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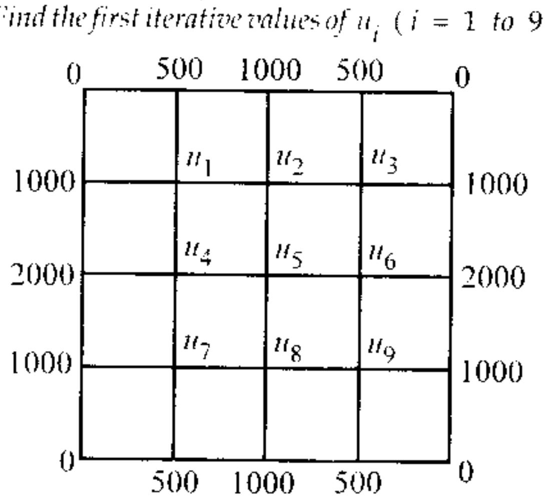
19. *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown. Find the first iterative values of u_i ($i = 1$ to 9) to the nearest integer.*



>> u_5 is located at the centre of the region and hence by the standard five point formula,

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$$

Next we shall compute u_1, u_3, u_7, u_9 by the diagonal five point formula .

$$u_1 = \frac{1}{4}(0 + 1500 + 2000 + 1000) = 1125$$

$$u_3 = \frac{1}{4}(1000 + 2000 + 1500 + 0) = 1125$$

Also $\quad u_7 = 1125 = u_9$

Further we compute u_2, u_4, u_6, u_8 by S.F.

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1187.5$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1437.5$$

$$u_6 = \frac{1}{4}(1500 + 2000 + 1125 + 1125) = 1437.5$$

$$u_8 = \frac{1}{4}(1125 + 1125 + 1500 + 1000) = 1187.5$$

These values are regarded as the initial approximations to commence the Liebmann's iteration. We compute u_i ($i = 1$ to 9) in the serial order by using the latest iterative values on hand by applying S.F.

First iteration

$u_1^{(1)} = \frac{1}{4}$ [1000 + 1187.5 + 500 + 1437.5] = 1031.25

$u_2^{(1)} = \frac{1}{4}$ [1031.25 + 1125 + 1000 + 1500] = 1164.0625

$u_3^{(1)} = \frac{1}{4}$ [1164.0625 + 1000 + 500 + 1437.5] = 1025.3906

$u_4^{(1)} = \frac{1}{4}$ [2000 + 1500 + 1031.25 + 1125] = 1414.0625

$u_5^{(1)} = \frac{1}{4}$ [1414.0625 + 1437.5 + 1164.0625 + 1187.5] = 1300.7813

$u_6^{(1)} = \frac{1}{4}$ [1300.78 + 2000 + 1025.3906 + 1125] = 1362.7930

$u_7^{(1)} = \frac{1}{4}$ [1000 + 1187.5 + 1414.0625 + 500] = 1025.3906

$u_8^{(1)} = \frac{1}{4}$ [1025.4 + 1125 + 1300.7930 + 1000] = 1112.7930

$u_9^{(1)} = \frac{1}{4}$ [1112.7930 + 1000 + 1362.7930 + 500] = 993.8965

Thus the required first iterative values to the nearest integer are as follows.

u_1 = **1031**, u_2 = **1164**, u_3 = **1025**, u_4 = **1414**

u_5 = **1301**, u_6 = **1363**, u_7 = **1025**, u_8 = **1113**, u_9 = **994**

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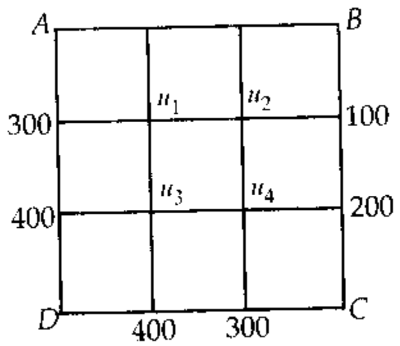
20. *Solve the elliptic eqation $u_{xx} + u_{yy} = 0$ at the pivotal points for the following square mesh using Liebman's method.*



>> It is evident that the values at the two boundary points on *AB* are respectively 200 and 100. We have by Liebman's method,

$u_1 = \frac{1}{4}(300 + u_2 + 200 + u_3)$ $\quad u_2 = \frac{1}{4}(u_1 + 100 + 100 + u_4)$

$u_3 = \frac{1}{4}(400 + u_4 + u_1 + 400)$ $\quad u_4 = \frac{1}{4}(u_3 + 200 + u_2 + 300)$

That is,

$$4u_1 - u_2 - u_3 = 500 \quad \ldots (1)$$

$$-u_1 + 4u_2 - u_4 = 200 \quad \ldots (2)$$

$$-u_1 + 4u_3 - u_4 = 800 \quad \ldots (3)$$

$$-u_2 - u_3 + 4u_4 = 500 \quad \ldots (4)$$

Thus by solving these (*as before*) we obtain the values at the pivotal points.

$$\mathbf{u_1 = 250,\ u_2 = 175,\ u_3 = 325,\ u_4 = 250}$$

EXERCISES

1. Solve : $u_{tt} = 16u_{xx}$ taking $h = 1$ in five steps given that $u(0, t) = 0$, $u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5-x)$

2. Solve : $u_{xx} = 0.04\,u_{tt}$ at the mesh points given that $u(0, t) = 0 = u(5, t)$, $u_t(x, 0) = 0$ and
$$u(x, 0) = \begin{cases} 2x, & 0 \le x \le 5/2 \\ 10 - 2x, & 5/2 \le x \le 5 \end{cases} \quad \text{(Take } h = 1\text{)}$$
compute $u(x, t)$ for $0 \le t \le 1$.

3. Solve : $u_{xx} = u_{tt}$ given that $u(0, t) = 0 = u(4, t)$, $u_t(x, 0) = 0$ and $u(x, 0) = \dfrac{x(4-x)}{2}$ by taking $h = 1$. Find $u(x, t)$ for $0 \le t \le 5$.

4. Solve : $u_{xx} = u_t$ subject to the conditions $u(0, t) = 0 = u(5, t)$ and $u(x, 0) = x^2(25 - x^2)$ by taking $h = 1$. Carryout 5 steps.

5. Find the values of $u(x, t)$ satisfying the parabolic equation $u_t = 4\,u_{xx}$ subject to the conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = \dfrac{x}{2}(8-x)$ at the points $x = i$, $i = 0, 1, 2, 3, \cdots 8$, $t = j$, $j = 0, 1, 2$

6. Using Schmidt's formula solve : $u_{xx} = 2u_t$ with the conditions $u(0, t) = 0$, $u(12, t) = 9$ for $0 \le t \le 12$ and $u(x, 0) = \dfrac{x}{4}(15 - x)$ for $0 \le x \le 12$ taking $h = 3 = k$

7. Solve : $u_{xx} + u_{yy} = 0$ over the square region of side 4 units subjected to the following boundary conditions :

(i) $u(0, y) = 0$ for $0 \le y \le 4$

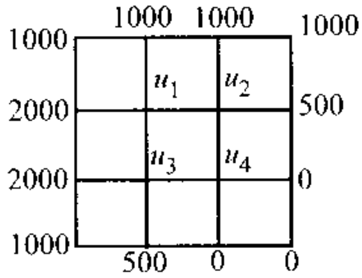
(ii) $u(4, y) = 8 + 2y$ for $0 \le y \le 4$

(iii) $u(x, 0) = x^2/2$ for $0 \le x \le 4$

(iv) $u(x, 4) = x^2$ for $0 \le x \le 4$

Perform two Liebmann's iterations.

8. Evaluate $u(x, y)$ satisfying Laplace's equation in two dimensions at the lattice points given the values on the boundary as indicated in the following figure



(Keep the answer to the nearest integer)

ANSWERS

1.

t \ x	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	−2	0
3	0	2	−2	−8	−7	0
4	0	−9	−14	−11	−6	0
5	0	−16	−18	−12	−4	0

2.

t \ x	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	−1	−1	−1	−1	0
4/5	0	−2	−3	−3	−2	0
1	0	−2	−4	−4	−2	0

3.

t \ x	0	1	2	3	4
0	0	1.5	2	1.5	0
1	0	1	1.5	1	0
2	0	0	0	0	0
3	0	− 1	− 1.5	− 1	0
4	0	− 1.5	− 2	− 1.5	0
5	0	− 1	− 1.5	− 1	0

4.

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
1/2	0	42	84	114	72	0
1	0	42	78	78	57	0
3/2	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0

5.

t \ x	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0

6.

t \ x	0	3	6	9	12
0	0	9	13.5	13.5	9
3	0	8.25	12.75	12.75	9
6	0	7.625	12	12.125	9
9	0	7.083	11.292	11.583	9
12	0	6.604	10.639	11.104	9

7. $u_1 = 2,\ u_2 = 4.92,\ u_3 = 9,\ u_4 = 2.07$

$u_5 = 4.69,\ u_6 = 8.07,\ u_7 = 1.57,\ u_8 = 3.71,\ u_9 = 6.57$

8. $u_1 = 1208,\ u_2 = 792,\ u_3 = 1042,\ u_4 = 458$